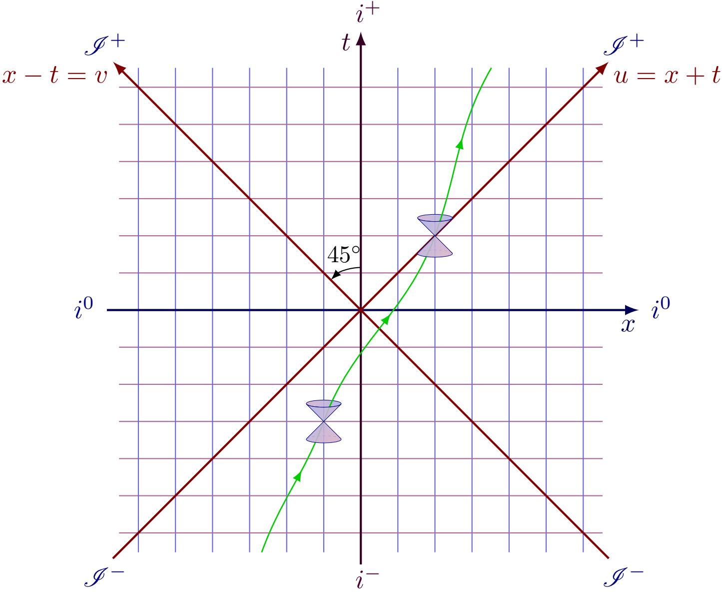
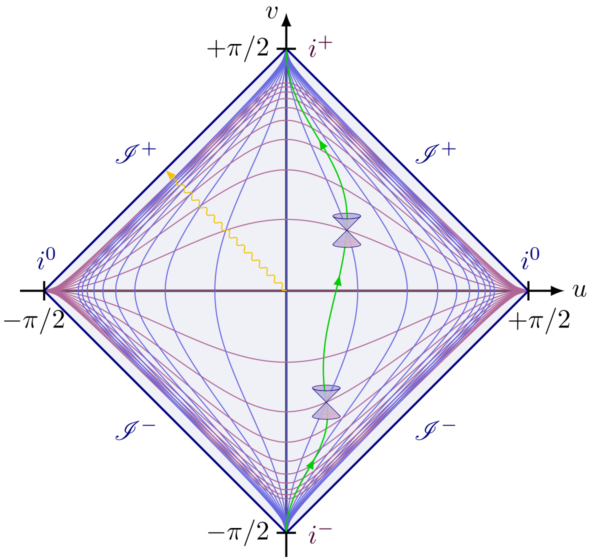


i^+
t
$\mathscr{I}^+$
$\mathscr{I}^+$
$x - t = v$
$u = x + t$
$45°$
i^0
i^0
x
$\mathscr{I}^-$
i^-
$\mathscr{I}^-$

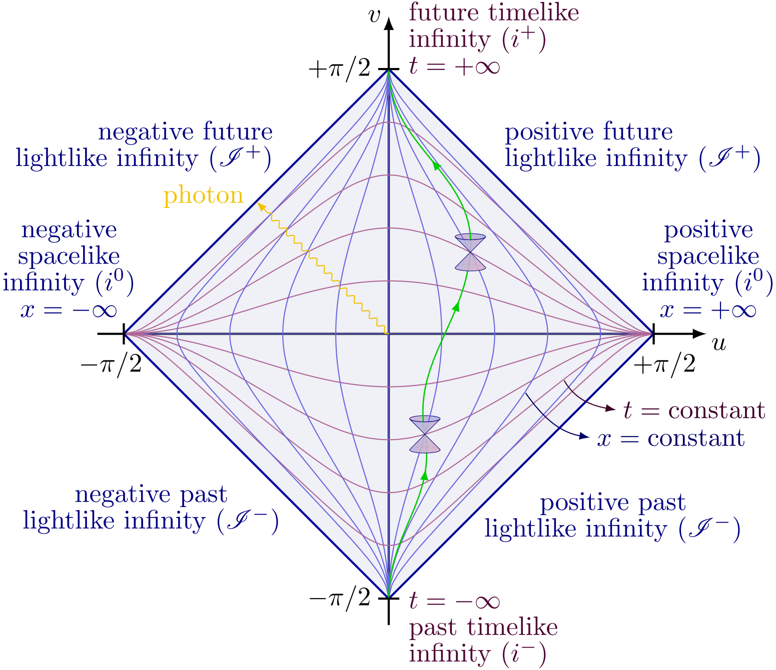
Penrose coordinates for Minkowski spacetime

$$\begin{cases} r + t = \tan(u + v) \\ r - t = \tan(u - v) \end{cases}$$

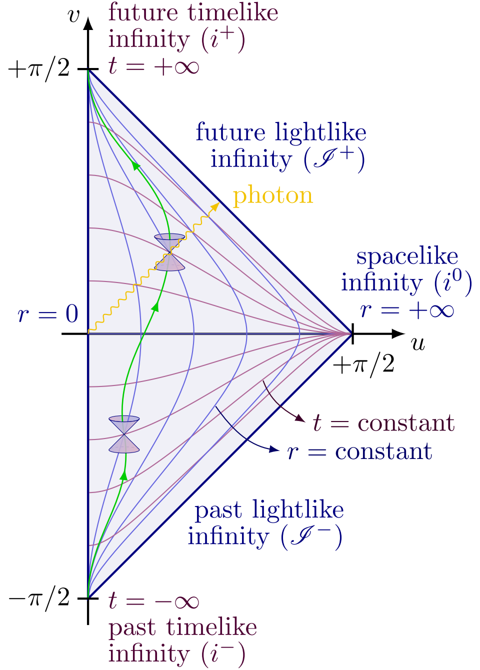
$$\Rightarrow \begin{cases} \tan v = \dfrac{\sqrt{(1+\tan^2 u)^2 + 4r^2 \tan^2 u} - (1+\tan^2 u)}{2r\tan^2 u} & \text{for constant } r \\ \\ \tan u = \dfrac{\sqrt{(1+\tan^2 v)^2 + 4t^2 \tan^2 v} - (1+\tan^2 v)}{2t\tan^2 v} & \text{for constant } t \end{cases}$$



v
$+\pi/2$
i^+
$\mathscr{I}^+$
$\mathscr{I}^+$
i^0
i^0
u
$-\pi/2$
$+\pi/2$
$\mathscr{I}^-$
$\mathscr{I}^-$
$-\pi/2$
i^-



v
future timelike infinity (i^+)
$t = +\infty$
$+\pi/2$
negative future lightlike infinity ($\mathscr{I}^+$)
positive future lightlike infinity ($\mathscr{I}^+$)
photon
negative spacelike infinity (i^0)
$x = -\infty$
positive spacelike infinity (i^0)
$x = +\infty$
$-\pi/2$
$+\pi/2$
u
$t =$ constant
$x =$ constant
negative past lightlike infinity ($\mathscr{I}^-$)
positive past lightlike infinity ($\mathscr{I}^-$)
$-\pi/2$
$t = -\infty$
past timelike infinity (i^-)



v
future timelike infinity (i^+)
$+\pi/2$
$t = +\infty$
future lightlike infinity ($\mathscr{I}^+$)
photon
spacelike infinity (i^0)
$r = +\infty$
$r = 0$
u
$+\pi/2$
$t = $ constant
$r = $ constant
past lightlike infinity ($\mathscr{I}^-$)
$-\pi/2$
$t = -\infty$
past timelike infinity (i^-)

Kruskal-Szekeres coordinates for Schwarzschild spacetime

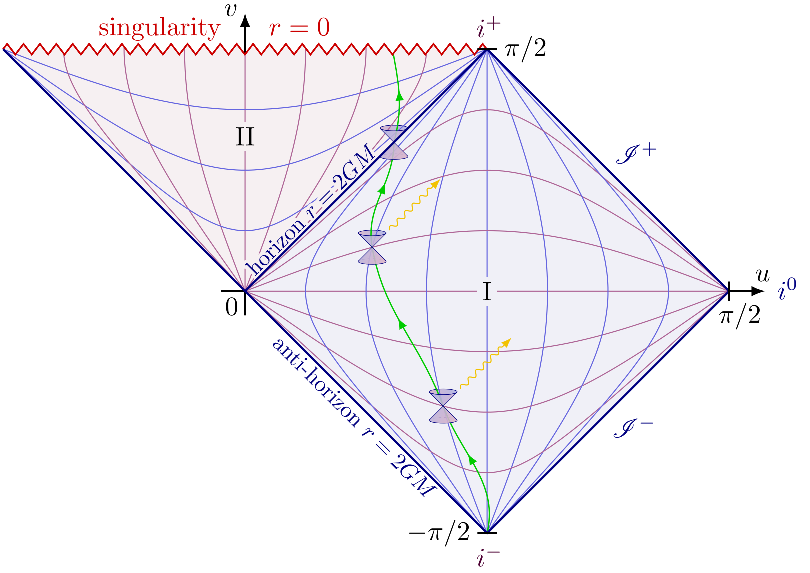
$$\begin{cases} \left.\begin{aligned} U &= \sqrt{\frac{r}{2GM}-1}\, e^{\frac{r}{4GM}} \cosh\left(\frac{t}{4GM}\right) \\ V &= \sqrt{\frac{r}{2GM}-1}\, e^{\frac{r}{4GM}} \sinh\left(\frac{t}{4GM}\right) \end{aligned}\right\} & \text{for } r > 2GM \\ \left.\begin{aligned} U &= \sqrt{1-\frac{r}{2GM}}\, e^{\frac{r}{4GM}} \sinh\left(\frac{t}{4GM}\right) \\ V &= \sqrt{1-\frac{r}{2GM}}\, e^{\frac{r}{4GM}} \cosh\left(\frac{t}{4GM}\right) \end{aligned}\right\} & \text{for } r < 2GM \end{cases}$$

$$\Rightarrow \begin{cases} U^2 - V^2 = \left(\frac{r}{2GM} - 1\right) e^{\frac{r}{2GM}} \\ V = \tanh\left(\frac{t}{4GM}\right) U & \text{for } r > 2GM \\ V = \coth\left(\frac{t}{4GM}\right) U & \text{for } r < 2GM \end{cases}$$

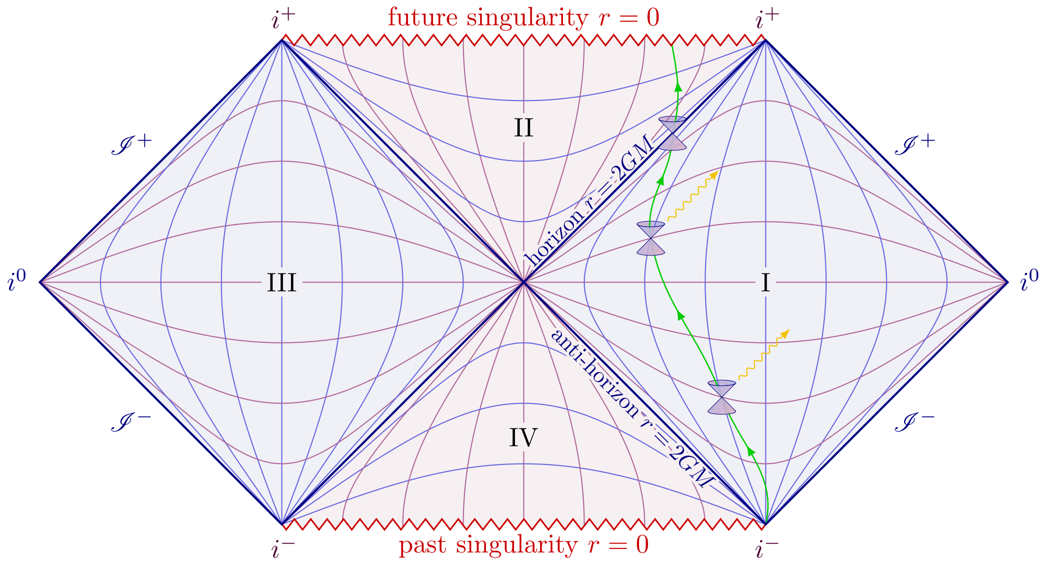
Penrose coordinates for Schwarzschild spacetime

$$\begin{cases} U + V = \tan(u+v) \\ U - V = \tan(u-v) \end{cases}$$

$$\Rightarrow \begin{cases} \cos(2v) = \dfrac{2GM + (r-2GM)\, e^{\frac{r}{2GM}}}{2GM - (r-2GM)\, e^{\frac{r}{2GM}}} \cos(2u) \\ \sin(2v) = \tanh\left(\dfrac{t}{4GM}\right) \sin(2u) & \text{for } r > 2GM \\ \sin(2v) = \cosh\left(\dfrac{t}{4GM}\right) \sin(2u) & \text{for } r < 2GM \end{cases}$$



v
singularity
$r = 0$
i^+
$\pi/2$
II
horizon $r = 2GM$
$\mathscr{I}^+$
I
u
i^0
0
$\pi/2$
anti-horizon $r = 2GM$
$\mathscr{I}^-$
$-\pi/2$
i^-



i^+
future singularity $r = 0$
i^+
II
$\mathscr{I}^+$
$\mathscr{I}^+$
horizon $r = 2GM$
i^0
III
I
i^0
anti-horizon $r = 2GM$
$\mathscr{I}^-$
$\mathscr{I}^-$
IV
i^-
past singularity $r = 0$
i^-